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Inhomogeneous Inverse Differential Realization of Two-Parameter Deformed Quasi-SU(1,1)$_{q,s}$ Group

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The generators and irreducible-representation and coherent state of the two-parameter deformed (q, s-deformed) quasi-SU(1,1)$_{q,s}$ group are constructed by using the inverse operators of the q, s-deformed bosonic oscillator, and the inhomogeneous inverse differential realization of the q, s-deformed quasi-SU(1,1)$_{q,s}$ group is derived.

The boson realization approach is very effective for studying the representation of groups, and the boson realization usually can be obtained from the creation and annihilation operators of the bosonic oscillator, as in the Jordan–Schwinger realization, etc. The nature of the inverse operator of the bosonic oscillator (Dirac, 1966) has also been studied, and some new results have been given (Mehta *et al.*, 1992; Fan, 1993, 1994). On the basis of Yu and Liu (1997, 1998, 1999), the present paper studies the inhomogeneous inverse differential realization of the q, s-deformed quasi-SU(1,1)$_{q,s}$ group by using the inverse operator of the q, s-deformed bosonic oscillator.

We first introduce four independent q, s-deformed bosonic oscillators as follows (Jing, 1993; Jing and Cuypers, 1993):

$$\begin{aligned} a_i^+ a_i &= [n_i^a]_{qs}, \qquad a_i a_i^+ = [n_i^a + 1]_{qs}, \\ [n_i^a, a_i^+] &= a_i^+, \qquad [n_i^a, a_i] = -a_i \qquad (i = 1, 2) \\ b_i^+ b_i &= [n_i^b]_{qs^{-1}}, \qquad b_i b_i^+ = [n_i^b + 1]_{qs^{-1}}, \end{aligned} \tag{1}$$

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$$[n_i^b, b_i^+] = b_i^+, \qquad [n_i^b, b_i] = -b_i \qquad (i = 1, 2) \tag{2}$$

They satisfy the following commutative relations:

$$a_i a_i^+ - s^{-1}q\, a_i^+ a_i = (sq)^{-n_i^a},$$
$$a_i a_i^+ - (sq)^{-1}\, a_i^+ a_i = (s^{-1}q)^{n_i^a} \qquad (i = 1, 2) \tag{3}$$

$$b_i b_i^+ - sq\, b_i^+ b_i = (sq^{-1})^{n_i^b} \qquad (i = 1, 2) \tag{4}$$

where we have used the notations $[x]_{qs} = s^{1-x}[x] = s^{1-x}(q^x - q^{-x})/(q - q^{-1})$ and $[x]_{qs^{-1}} = s^{x-1}\,[x]$.

Similar to Fan's work (Fan, 1994) and our recent result (Yu and Liu, 1999), it is easy to get the inverses of the operators a_i, a_i^+ and b_i, b_i^+ as follows:

$$a_i^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n+1]_{qs}}} \big|n+1\rangle\langle n\big|,$$
$$b_i^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n+1]_{qs^{-1}}}} \big|n+1\rangle\langle n\big| \tag{5}$$

$$(a_i^+)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n+1]_{qs}}} \big|n\rangle\langle n+1\big|,$$
$$(b_i^+)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n+1]_{qs^{-1}}}} \big|n\rangle\langle n+1\big| \tag{6}$$

we get

$$a_i a_i^{-1} = (a_i^+)^{-1} a_i^+ = 1, \qquad b_i b_i^{-1} = (b_i^+)^{-1} b_i^+ = 1 \tag{7}$$

$$a_i^{-1} a_i = a_i^+ (a_i^+)^{-1} = 1 - \big|\mathrm{o}\rangle_{aa}\langle \mathrm{o}\big|, \qquad b_i^{-1} b_i = b_i^+ (b_i^+)^{-1} = 1 - \big|\mathrm{o}\rangle_{bb}\langle \mathrm{o}\big| \tag{8}$$

where the vacuum state projection operators $\big|\mathrm{o}\rangle_{aa}\langle \mathrm{o}\big|$ and $\big|\mathrm{o}\rangle_{bb}\langle \mathrm{o}\big|$ can be obtained from $a_i\big|\mathrm{o}\rangle_{aa}\langle \mathrm{o}\big| = 0$ and $b_i\big|\mathrm{o}\rangle_{bb}\langle \mathrm{o}\big| = 0$, namely (Yu *et al.*, 1996)

$$\big|\mathrm{o}\rangle_{aa}\langle \mathrm{o}\big| = \sum_{n=0}^{\infty} \frac{(-1)^n (a_i^+)^n\, (sq)^{n(n_i^a+n-1)} a_i^n}{[n]_{qs^{-1}}!} \qquad (i = 1, 2) \tag{9}$$

$$\big|\mathrm{o}\rangle_{bb}\langle \mathrm{o}\big| = \sum_{n=0}^{\infty} \frac{(-1)^n (b_i^+)^n\, (s^{-1}q)^{n(n_i^b+n-1)} b_i^n}{[n]_{qs}!} \qquad (i = 1, 2) \tag{10}$$

The q,s-deformed $SU(1,1)_{q,s}$ group has three types of unitary irreducible representations (Jing and Cuypers, 1993): a positive discrete series (a), a negative discrete series (b), and a continuous series which we do not consider here. The generators of the q,s-deformed $SU(1,1)_{q,s}$ group can be obtained from the combination of inverse operators:

$$(J_+^a)^{-1} = (a_1^+)^{-1}(a_2^+)^{-1}, \qquad (J_-^a)^{-1} = a_1^{-1}a_2^{-1} \tag{11}$$

$$(J_0^a)^{-1} = \frac{1}{2}\{(N_1^a)^{-1}(N_2^a)^{-1} - (N_1^a + 1)^{-1}(N_2^a + 1)^{-1}\} \tag{12}$$

and

$$(J_+^b)^{-1} = b_2^{-1}b_1^{-1}, \qquad (J_-^b)^{-1} = (b_1^+)^{-1}(b_2^+)^{-1} \tag{13}$$

$$(J_0^b)^{-1} = -\frac{1}{2}\{(N_1^b)^{-1}(N_2^b)^{-1} - (N_1^b + 1)^{-1}(N_2^b + 1)^{-1}\} \tag{14}$$

where we have defined the following relations:

$$(N_i^a)^{-1} = a_i^{-1}(a_i^+)^{-1}, \qquad (N_i^a + 1)^{-1} = (a_i^+)^{-1}a_i^{-1} \qquad (i = 1, 2) \tag{15}$$

$$(N_i^b)^{-1} = b_i^{-1}(b_i^+)^{-1}, \qquad (N_i^b + 1)^{-1} = (b_i^+)^{-1}b_i^{-1} \qquad (i = 1, 2) \tag{16}$$

It is easy to find that

$$s^{-1}(J_+^{a(b)})^{-1}(J_-^{a(b)})^{-1} - s(J_-^{a(b)})^{-1}(J_+^{a(b)})^{-1} = -s^{-2(J_0^{a(b)})^{-1}}[2(J_0^{a(b)})^{-1}] \tag{17}$$

Therefore a nonclosed q,s-deformed SU(1,1)$_{q,s}$ group is constructed; we call it the quasi-SU(1,1)$_{q,s}$ group.

The two discrete unitary irreducible representations $|l, r\rangle^a$ and $|l, r\rangle^b$ of the SU(1,1)$_{q,s}$ group are, respectively,

$$|l, r\rangle^a = |r - l - 1\rangle^a \otimes |r + l\rangle^a \qquad (r \geq -l > 0) \tag{18}$$

$$|l, r\rangle^b = |-r - l - 1\rangle^b \otimes |-r + l\rangle^b \qquad (r \leq l < 0) \tag{19}$$

These irreducible representations are infinite dimensional and depend on the quantum numbers $l = -1/2, -1, \ldots$. The action of the quasi-SU(1,1)$_{a,s}$ group generators on the elements of the irreducible representations (18) and (19) is given by

$$(J_+^a)^{-1}|l, r\rangle^a = \frac{1}{s\sqrt{[r - l - 1]_{qs}[r + l]_{qs}}}|l, r - 1\rangle^a \tag{20}$$

$$(J_-^a)^{-1}|l, r\rangle^a = \frac{1}{s\sqrt{[r - l]_{qs}[r + l + 1]_{qs}}}|l, r + 1\rangle^a \tag{21}$$

$$(J_0^a)^{-1}|l, r\rangle^a = \frac{[r - l]_{qs}[r + l + 1]_{qs} - [r + l]_{qs}[r - l - 1]_{qs}}{2[r + l]_{qs}\,[r + l + 1]_{qs}[r - l]_{qs}\,[r - l - 1]_{qs}}|l, r\rangle^a \tag{22}$$

and

$$(J_+^b)^{-1}|l, r\rangle^b$$

$$= \frac{s}{\sqrt{[-r - l]_{qs^{-1}}[-r + l + 1]_{qs^{-1}}}}|l, r - 1\rangle^b \tag{23}$$

$$(J_-^b)^{-1}\left|l, r\right\rangle^b$$

$$= \frac{s}{\sqrt{[-r + l]_{qs^{-1}}[-r - l - 1]_{qs^{-1}}}}\left|l, r + 1\right\rangle^b \tag{24}$$

$$(J_0^b)^{-1}\left|l, r\right\rangle^b$$

$$= \frac{[-r - l - 1]_{qs^{-1}}[-r + l]_{qs^{-1}} - [-r - l]_{qs^{-1}}[-r + l + 1]_{qs^{-1}}}{2[-r - l - 1]_{qs^{-1}}[-r + l]_{qs^{-1}}[-r - l]_{qs^{-1}}[-r + l + 1]_{qs^{-1}}} \tag{25}$$

The coherent states of the irreducible representation for the quasi-$SU(1,1)_{q,s}$ group corresponding to the positive discrete series (a) and the negative discrete series (b) are, respectively,

$$\left|lz\right)^a = e^{z(J_-^a)^{-1}}\left|l, -l\right\rangle^a$$

$$= \sum_{r=-l}^{\infty} \frac{1}{(l + r)!}\sqrt{\frac{[-2l - 1]_{qs}!}{[l + r]_{qs}!\,[r - l - 1]_{qs}!}}(s^{-1}z)^{l+r}\left|l, r\right\rangle^a \tag{26}$$

$$\left|lz\right)^b = e^{z(J_+^b)^{-1}}\left|l, l\right\rangle^b$$

$$= \sum_{r=l}^{-\infty} \frac{1}{(l - r)!}\sqrt{\frac{[-2l - 1]_{qs^{-1}}!}{[l - r]_{qs^{-1}}!\,[-r - l - 1]_{qs^{-1}}!}}(sz)^{l-r}\left|l, r\right\rangle^b \tag{27}$$

Their normalization coefficients are

$$M_l^a(|z|^2) = \sum_{r=-l}^{\infty} \frac{[-2l - 1]_{qs}!}{((l + r)!)^2\,[l + r]_{qs}!\,[r - l - 1]_{qs}!}(|s^{-1}z|^2)^{l+r} \tag{28}$$

$$M_l^b(|z|^2) = \sum_{r=l}^{-\infty} \frac{[-2l - 1]_{qs^{-1}}!}{((l - r)!)^2\,[l - r]_{qs^-}!\,[-r - l - 1]_{qs^{-1}}!}(|sz|^2)^{l-r} \tag{29}$$

Using the method proposed by Yu *et al.* (1997a, b), we can obtain the completeness relations of the quantum states $\left|lz\right)^a$ and $\left|lz\right)^b$ as follows:

$$\frac{1}{\pi}(\rho^a)^{-1}\int \frac{\left|lz\right)^{aa}\left(lz\right|}{M_l^a(|z|^2)}dz^2 = 1, \qquad \frac{1}{\pi}(\rho^b)^{-1}\int \frac{\left|lz\right)^{bb}\left(lz\right|}{M_l^b(|z|^2)}dz^2 = 1 \tag{30}$$

We now define two state vectors in the space of the irreducible representation

$$\left|\psi\right\rangle^a = \sum_{r=-l}^{\infty} C_r^a\left|l, r\right\rangle^a, \qquad \left|\psi\right\rangle^b = \sum_{r=l}^{-\infty} C_r^b\left|l, r\right\rangle^b \tag{31}$$

We get

$$\frac{s^{-2}}{[-2l + z\, d/dz]_{qs}(1/z)(z\, d/dz)[z\, d/dz]_{qs}}\, {}^{a}\langle l\bar{z}|\psi\rangle^{a}$$
$$= \sum_{r=-l}^{\infty} \frac{C_r^a}{(l+r+1)!\,[l+r+1]_{qs}\,[r-l]_{qs}} \times \sqrt{\frac{[-2l-1]_{qs}!}{[r+l]_{qs}!\,[r-l-1]_{qs}!}}\, s^{-1}(s^{-1}z)^{l+r+1} \tag{32}$$

$$\frac{s^{2}}{[-2l + z\, d/dz]_{qs^{-1}}\,(1/z)\,(z\, d/dz)\,[z\, d/dz]_{qs^{-1}}}\, {}^{b}\langle l\bar{z}|\psi\rangle^{b}$$
$$= \sum_{r=l}^{-\infty} \frac{C_r^b}{(l-r+1)!\,[l-r+1]_{qs^{-1}}\,[-r-l]_{qs^{-1}}} \times \sqrt{\frac{[-2l-1]_{qs^{-1}}!}{[l-r]_{qs^{-1}}!\,[-r-l-1]_{qs^{-1}}!}}\, s(sz)^{l-r+1} \tag{33}$$

On the other hand, we also have

$${}^{a}\langle l\bar{z}|(J_-^a)^{-1}|\psi\rangle^{a} = \sum_{r=-l}^{\infty} C_r^a\, {}^{a}\langle l\bar{z}|(J_-^a)^{-1}|l, r\rangle^{a}$$
$$= \sum_{r=-l}^{\infty} \frac{C_r^a}{(l+r+1)!\,[l+r+1]_{qs}\,[r-l]_{qs}} \times \sqrt{\frac{[-2l-1]_{qs}!}{[r+l]_{qs}!\,[r-l-1]_{qs}!}}\, s^{-1}\,(s^{-1}z)^{l+r+1} \tag{34}$$

$${}^{b}\langle l\bar{z}|(J_+^b)^{-1}|\psi\rangle^{b} = \sum_{r=l}^{-\infty} C_r^b\, {}^{b}\langle l\bar{z}|(J_+^b)^{-1}|l, r\rangle^{b}$$
$$= \sum_{r=l}^{-\infty} \frac{C_r^b}{(l-r+1)!\,[l-r+1]_{qs^{-1}}[-r-l]_{qs^{-1}}} \times \sqrt{\frac{[-2l-1]_{qs^{-1}}!}{[l-r]_{qs^{-1}}!\,[-r-l-1]_{qs^{-1}}}}\, s(sz)^{l-r+1} \tag{35}$$

From Eqs. (32)–(35), we have the inhomogeneous inverse differential realization of the operators $(J_-^a)^{-1}$ and $(J_+^b)^{-1}$):

$$B_l^a((J_-^a)^{-1}) = \frac{s^{-2}}{[-2l + z\, d/dz]_{qs}\, (1/z)\, (z\, d/dz)\, [z\, d/dz]_{qs}} \tag{36}$$

$$B_l^b((J_+^b)^{-1}) = \frac{s^{2}}{[-2l + z\, d/dz]_{qs^{-1}}\, (1/z)\, (z\, d/dz)\, [z\, d/dz]_{qs^{-1}}} \tag{37}$$

Similarly, we can obtain

$$\begin{aligned} {}^a(l\bar{z}|(J_+^a)^{-1}|\psi\rangle^a &= \sum_{r=-l}^{\infty} \frac{C_r^a}{(l+r-1)!} \sqrt{\frac{[-2l-1]_{qs}!}{[l+r]_{qs}!\,[r-l-1]_{qs}!}}\, s^{-1}(s^{-1}z)^{l+r-1} \\ &= \frac{d}{dz}\, {}^a(l\bar{z}|\psi\rangle^a \end{aligned} \tag{38}$$

$$\begin{aligned} {}^a(l\bar{z}|(J_o^a)^{-1}|\psi\rangle^a &= \sum_{r=-l}^{\infty} \frac{C_r^a\,([r-l]_{qs}[r+l+1]_{qs} - [r+l]_{qs}[r-l-1]_{qs})}{2(l+r)!\,[r+l]_{qs}[r+l+1]_{qs}[r-l]_{qs}[r-l-1]_{qs}} \\ &\quad \times \sqrt{\frac{[-2l-1]_{qs}!}{[l+r]_{qs}!\,[r-l-1]_{qs}!}}\,(s^{-1}z)^{l+r} \\ &= \frac{1}{2}\{[z\,d/dz - 2l]_{qs}[z\,d/dz + 1]_{qs} - [z\,d/dz]_{qs}[z\,d/dz - 2l - 1]_{qs}\} \\ &\quad \times \frac{1}{z[-2l + z\,d/dz]_{qs}(1/z)\,[-2l + z\,d/dz]_{qs}(1/z)\,[z\,d/dz]_{qs}(z)\,[z\,d/dz]_{qs}} \\ &\quad \times {}^a(l\bar{z}|\psi\rangle^a \end{aligned} \tag{39}$$

$$\begin{aligned} {}^b(l\bar{z}|(J_-^b)^{-1}|\psi\rangle^b &= \sum_{r=l}^{-\infty} \frac{c_r^b}{(l-r-1)!} \sqrt{\frac{[-2l-1]_{qs^{-1}}!}{[l-r]_{qs^{-1}}!\,[-r-l-1]_{qs^{-1}}!}}\, s(sz)^{l-r-1} \\ &= d/dz\, {}^b(l\bar{z}|\psi\rangle^b \end{aligned} \tag{40}$$

$$\begin{aligned} {}^b(l\bar{z}|(J_0^b)^{-1}|\psi\rangle^b &= \sum_{r=l}^{-\infty} \frac{C_r^b\,([-r-l-1]_{qs^{-1}}[-r+l]_{qs^{-1}} - [-r-l]_{qs^{-1}}[-r+l+1]_{qs^{-1}})}{2(l-r)!\,[-r-l-1]_{qs^{-1}}[-r+l]_{qs^{-1}}[-r-l]_{qs^{-1}}[-r+l+1]_{qs^{-1}}} \\ &\quad \times \sqrt{\frac{[-2l-1]_{qs^{-1}}!}{[l-r]_{qs^{-1}}!\,[-r-l-1]_{qs^{-1}}!}}\,(sz)^{l-r} \end{aligned}$$

$$= \frac{1}{2}\{[-2l + z\, d/dz - 1]_{qs^{-1}}[z\, d/dz]_{qs^{-1}} - [-2l + z\, d/dz]_{qs^{-1}}[z\, d/dz + 1]_{qs^{-1}}\} \times \frac{1}{z\,[z\, d/dz - 2l]_{qs^{-1}}\,(1/z^2)\,[z\, d/dz]_{qs^{-1}}(z)\,[z\, d/dz]_{qs^{-1}}[-2l + z\, d/dz]_{qs^{-1}}} \times {}^{b}(l\bar{z}|\psi\rangle^{b} \tag{41}$$

Therefore we get the inhomogeneous inverse differential realizations of $(J_+^a)^{-1}$, $(J_o^a)^{-1}$, $(J_-^b)^{-1}$, and $(J_o^b)^{-1}$ in Bargmann space, respectively:

$$B_l^a((J_+^a)^{-1}) = d/dz \tag{42}$$

$$B_l^a((J_o^a)^{-1}) = \frac{1}{2}\{[z\, d/dz - 2l]_{qs}[z\, d/dz + 1]_{qs} - [z\, d/dz]_{qs}[z\, d/dz - 2l - 1]_{qs}\} \times \frac{1}{z[-2l + z\, d/dz]_{qs}(1/z)[-2l + z\, d/dz]_{qs}(1/z)[z\, d/dz]_{qs}(z)\,[z\, d/dz]_{qs}} \tag{43}$$

$$B_l^b((J_-^b)^{-1}) = d/dz \tag{44}$$

$$B_l^b((J_o^b)^{-1}) = \frac{1}{2}\{[-2l + z\, d/dz - 1]_{qs^{-1}}[z\, d/dz]_{qs^{-1}} - [-2l + z\, d/dz]_{qs^{-1}}[z\, d/dz + 1]_{qs^{-1}}\} \times \frac{1}{z[z\, d/dz - 2l]_{qs^{-1}}(1/z^2)\,[z\, d/dz]_{qs^{-1}}(z)\,[z\, d/dz]_{qs^{-1}}[-2l + z\, d/dz]_{qs^{-1}}} \tag{45}$$

From the above discussion, we conclude that Eqs. (36)–(37) and (42)–(45) are the inhomogeneous inverse differential realizations of the q, s-deformed quasi-SU$(1,\ 1)_{q,s}$ group.

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